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M.Sc. Sem-II Paper V (MAT 005)

Elementary Set-Theory (continue)

Injection:- The function $f: A \to B$ is an injection iff $(\forall a, a' \in A)$ if $a \neq a'$ then $f(a) \neq f(a')$

Surjection:- The function $f: A \to B$ is a surjection iff $(\forall b \in B)(\exists a \in A)$ such that $f(a) = b$.

Composition:- If $f: A \to B$ and $g: B \to C$ are functions, their composition $g \circ f : A \to C$ is dined by $(g \circ f)(x) = g(f(x))$

Theorem:- If $f: A \to B$ and $g: B \to C$ are surjections, then $g \circ f: A \to C$ is also a surjection.

Proof:- Let $c \in C$ be arbitrary.
Since g is surjective, $\exists\, b \in B$ such that $g(b) = c$.
Since f is surjective, $\exists\, a \in A$ such that $f(a) = b$.
Then, $(g \circ f)(a) = g(f(a)) = c$, hence $g \circ f$ is a surjection.

Bijection:- A function $f: A \to B$ is a bijection if f is an injection and a surjection.

Theorem:- If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Proof:- Composition of surjections is surjection and compositions of injections are injections.
So, $f: A \to B$ and $g: B \to C$ are bijections
Then $g \circ f : A \to C$ is a bijection.

<u>Inverse Function</u> :- If $f : A \to B$ is a bijection, then its inverse, $f^{-1} : B \to A$ is defined by $f^{-1}(b) =$ the unique $a \in A$ such that $f(a) = b$.

Note - If $f : A \to B$ is a bijection, it is easily checked that $f^{-1} : B \to A$ is a bijection.

In terms of ordered pairs, $f^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in f\}$

<u>Equinumerous</u> :- Two sets A and B, are equinumerous, written $A \sim B$ iff there exists a bijection $f : A \to B$.

<u>Theorem</u>:- Let $E = \{0, 2, 4, --\}$ be the even natural numbers. Then, $N \sim E$

<u>Proof</u>:- we can define a bijection $f : N \to E$ by $f(n) = 2n$

<u>Note</u> - If is often extremely difficult to explicity define a bijection $f : N \to A$. However, suppose that $f : N \to A$ is a bijection. For each $n \in N$ let a_n be $f(n)$. Then, $a_0, a_1, ---$ is a list of the elements of A such that every element occurs exactly once, and conversely, if such a list exists, then we can define a bijection $f : N \to A$ by $f(n) = a_n$.

<u>Theorem</u> :- $N \sim Z$

Proof - we can list the elements of $Z : 0, 1, -1, -2, 2, ---$

<u>Theorem</u> :- $N \sim Q$

Proof :- we proceed in two steps.

First, we prove that $N \sim Q^+ = \{q \in Q : q > 0\}$ and consider the following infinite matrix

$$\begin{array}{ccc}
\frac{1}{1} \rightarrow & \frac{2}{1} & \frac{3}{1} \rightarrow \\
\frac{1}{2} \swarrow & \frac{2}{2} \nearrow & \frac{3}{2} \swarrow \\
\downarrow \frac{1}{3} \nearrow & \frac{2}{3} \swarrow & \frac{3}{3}
\end{array}$$

Proceed through the matrix along the indicated route adding rational numbers, if they have not already occurred

Second, we declare the $N \sim Q$. In the first part, we showed that there exists a bijection $f : N \to Q^+$ hence we can list Q by $0, f(1), -f(1), ---$