

①

Dr. Kamlesh Kumar  
Asst. Prof. (Guest Faculty)  
Dept. of Mathematics  
Maharaja College,  
V.K.S.U, Agra

Date  
19/04/21

M.Sc. Sem-II Paper V (MAT 005)  
Elementary Set-Theory (Continue)

Injection:- The function  $f: A \rightarrow B$  is an injection iff  $(\forall a, a' \in A)$  if  $a \neq a'$  then  $f(a) \neq f(a')$

Surjection:- The function  $f: A \rightarrow B$  is a surjection iff  $(\forall b \in B) (\exists a \in A)$  such that  $f(a) = b$ .

Composition:- If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are functions, their composition  $g \circ f: A \rightarrow C$  is ~~defined~~ defined by  $(g \circ f)(x) = g(f(x))$

Theorem:- If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are surjections, then  $g \circ f: A \rightarrow C$  is also a surjection.

Proof:- Let  $c \in C$  be arbitrary.

Since  $g$  is surjective,  $\exists b \in B$  such that  $g(b) = c$ .

Since  $f$  is surjective,  $\exists a \in A$  such that  $f(a) = b$ .

Then,  $(g \circ f)(a) = g(f(a)) = c$ , hence  $g \circ f$  is a surjection.

Bijection:- A function  $f: A \rightarrow B$  is a bijection if  $f$  is an injection and a surjection.

Theorem:- If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are bijections, then  $g \circ f: A \rightarrow C$  is a bijection.

Proof:- Composition of surjections is surjection and compositions of injections are injections.

So,  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are bijections then  $g \circ f: A \rightarrow C$  is a bijection.



Inverse Function :- If  $f: A \rightarrow B$  is a bijection, then its inverse,  $f^{-1}: B \rightarrow A$  is defined by  $f^{-1}(b) =$  the unique  $a \in A$  such that  $f(a) = b$ .

Note - If  $f: A \rightarrow B$  is a bijection, it is easily checked that  $f^{-1}: B \rightarrow A$  is a bijection.

In terms of ordered pairs,  $f^{-1} = \{(b, a) : (a, b) \in f\}$

Equinumerous :- Two sets  $A$  and  $B$ , are equinumerous, written  $A \sim B$  iff there exists a bijection  $f: A \rightarrow B$ .

Theorem :- Let  $E = \{0, 2, 4, \dots\}$  be the even natural numbers. Then,  $N \sim E$

Proof :- we can define a bijection  $f: N \rightarrow E$  by  $f(n) = 2n$

Note - It is often extremely difficult to explicitly define a bijection  $f: N \rightarrow A$ . However, suppose that  $f: N \rightarrow A$  is a bijection. For each  $n \in N$  let  $a_n$  be  $f(n)$ . Then,  $a_0, a_1, \dots$  is a list of the elements of  $A$  such that every element occurs exactly once, and conversely, if such a list exists, then we can define a bijection  $f: N \rightarrow A$  by  $f(n) = a_n$ .

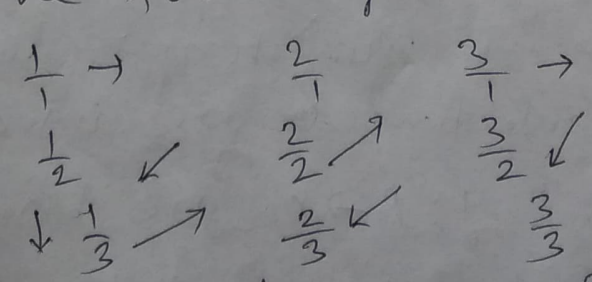
Theorem :-  $N \sim Z$

Proof - we can list the elements of  $Z: 0, 1, -1, 2, -2, \dots$

Theorem :-  $N \sim Q$

Proof :- we proceed in two steps.

First, we prove that  $N \sim Q^+ = \{q \in Q : q > 0\}$  and consider the following infinite matrix



Proceed through the matrix along the indicated route adding rational numbers, if they have not already occurred.

Second, we deduce the  $N \sim Q$ . In the first part, we showed that there exists a bijection  $f: N \rightarrow Q^+$  hence we can list  $Q$  by  $0, f(1), -f(1), \dots$